Central European Journal of Operations Research (2019) 27:1221–1244
https://doi.org/10.1007/s10100-018-0560-9

ORIGINAL PAPER

The network data envelopment analysis models for non-homogenous decision making units based on the sun network structure

Qingyou Yan[1] · Fei Zhao[1] · Xu Wang[1] · Guoliang Yang[2] · Tomas Baležentis[3] · Dalia Streimikiene[3]

Published online: 15 June 2018

Abstract
This paper seeks to propose a network data envelopment analysis (DEA) framework for analysis of heterogeneous systems. The paper introduces the dummy connector so that every network structure can be transformed into the sun network structure. In his case, the dummy connector allows for heterogeneity of the decision making units (DMUs) in terms of their inner structure. Based on the sun network structure, the static and dynamic network DEA models are established. Thus, DMUs with different structures can be evaluated according to the static and dynamic network DEA models. The efficiency of each sub-unit, each period and each sub-unit in each period can also be obtained. Two simulated examples are presented using the static and dynamic DEA models.

Keywords Data envelopment analysis (DEA) · Network DEA · Dynamic DEA · Dummy connector · Heterogeneous structures · Sun network structure

1 Introduction

Data envelopment analysis (DEA) is widely used to evaluate the performance of homogenous entities termed decision making units (DMUs). DEA, first proposed by Charnes et al. (1978), uses the linear programming to implement the Debreu-Farrell measure. In the early stage, the inner structure of the DEA model was considered as a black-box, and the development of DEA was mainly focused on the feasibility of

✉ Tomas Baležentis
tomas@laei.lt

1 School of Economics and Management, North China Electric Power University, Beijing, China

2 Institutes of Science and Development, Chinese Academy of Sciences, Beijing 100190, China

3 Lithuanian Institute of Agrarian Economics, Vilnius, Lithuania

the model. There have been different DEA models proposed to address various kinds of real-world problems (Emrouznejad et al. 2008; Cook and Seiford 2011; Song et al. 2018).

Indeed, many problems in the areas of business, environment and socioeconomic development require multi-period (dynamic) models involving complex structures represented by multiple conflicting objectives (variables), see Keshavarz et al. (2017), Kapelko (2018), Fang et al. (2018), Yin (2017), Toloo et al. (2018). The DEA has been extended to accommodate such complex problems by modelling the inner structure of the underlying technology. The network DEA is one of the approaches allowing to do so. Research on the network DEA has focused on models explicitly defining the internal structure of DMUs. The idea of the two-stage DEA system was first addressed by Charnes et al. (1986), and the two-stage DEA models have been developed rapidly both in theory and applications. For example, Seiford and Zhu (1999a) used the two-stage DEA model to measure the profitability and marketability of US commercial banks. Zhu (2000) applied the same two-stage DEA model to Fortune Global 500 companies. Cook et al. (2010) presented a review on the models of the basic two-stage system. Li et al. (2018) considered a two-stage DERA model where the information about the dominant stage is missing. Chen and Zhu (2018) looked into the determination of returns to scale in the two-stage DEA models. There have also been extensions of DEA allowing for shared inputs (Toloo et al. 2015) or outputs (Mahdiloo et al. 2018) according to different objectives associated with the activities modelled.

There have been a number of extensions presented for the two-stage DEA models: a series structure, a parallel structure, and a mixture of these. Jablonsky (2018) presented a two-stage DEA model for Olympics which allows considering the resources available, the resulting team size and the medals won which, indeed, illustrates the underlying logics of the serial models. The concept of the network DEA was proposed by Kao (2014). The network DEA models the operations for different sub-processes when measuring efficiency. Kao classified the network DEA models into nine types, i.e., the independent model by Zhu (2000), the system distance model by Chen et al. (2010), the process distance model by Chen and Zhu (2004), the factor distance model by Chen et al. (2013), the slacks-based measure model by Tone and Tsutsui (2009), the ratio-form system efficiency model by Chen et al. (2012), the ratio-form process efficiency (Chen et al. 2009; Cook et al. 2010; Lim and Zhu 2013), the game theoretic model by Du et al. (2011) and the value-based model by Wei and Chang (2011). Besides, Kao (2014) classified the five types of the internal structure of the network DEA model, including the basic two-stage structure, the general two-stage structure, the series structure, the parallel structure, and the dynamic structure.

The basic two-stage structure is the simplest network structure where all exogenous inputs enter the first process and are converted into intermediate inputs, all of which then serve as inputs to the second process, where the final outputs are produced (Chen and Zhu 2004; Guo et al. 2017). The general two-stage structure generalizes the basic two-stage structure by allowing both stages to consume exogenous inputs and to produce final outputs. The series structure consists of a series of process connected in a sequence (Park and Park 2009). With respect to the parallel structure, the most distinctive feature there is that all the processes in the parallel structure operate independently (Färe et al. 1997). The dynamic structure is used to solve the multi-period problems by interconnecting single-period systems together by the carry-over factors (Färe et al. 1996; Lv et al. 2017; Jablonsky 2016; Villa and Lozano 2018; Tran and Villano 2018).

All the network DEA models mentioned above can only be used to evaluate the efficiency of a set of homogenous DMUs. However, there exist certain production systems encompassing DMUs oriented towards production of the same outputs, yet featuring different structure. Therefore, there is a necessity to propose the network DEA model which can evaluate DMUs with different structures. Barat et al. (2018) proposed a network DEA model for non-homogenous DMUs. However, the non-homogeneous network DEA model for the dynamic setting has not been proposed. What is more, the inclusion of the undesirable outputs has not been discussed in the context of non-homogeneous dynamic network DEA.

In this paper, we introduce the dummy connector and develop the dynamic network DEA models involving undesirable outputs. By means of the dummy connector, all the network structures can be transformed into the sun structure. Based on this premise, we construct the static and dynamic DEA models to evaluate the efficiency of non-homogenous DMUs. This framework allows one to assess the efficiency of a certain sub-unit, a certain period and a certain sub-unit at a certain period.

The rest of the paper is organized as follows: The sun structure along with the static and dynamic DEA models based on the sun structure are presented in Sect. 2. The examples based on the static and dynamic DEA models are presented in Sect. 3. The conclusions are drawn in Sect. 4.

2 Methodology

We begin our disposition by introducing the key symbols and notations used in the paper. Let there be n DMUs. Each DMU comprises K sub-units. There are m inputs consumed, s outputs produced and D intermediate inputs forwarded in each DMU. Let us focus on an arbitrary DMU_o. We assume $X_o = (x_{1o}, \ldots, x_{mo})$, $Y_o = (y_{1o}, \ldots, y_{so})$ and $Z_o = (z_{1o}, \ldots, z_{Do})$ are the vectors of the inputs, outputs and intermediate inputs for DMU_o respectively. $I(k)$ indicates the set of inputs for the k-th sub-unit in DMU_o. Similarly, $O(k)$ indicates the set of outputs for the k-th sub-unit in DMU_o. $D^{in}(k)$ refers to the set of the intermediate inputs which are consumed by the k-th sub-unit in DMU_o. $D^{out}(k)$ refers to the set of the intermediate inputs which are produced by the k-th sub-unit in DMU_o. It is assumed that $D^{out}(k) \cap D^{in}(k) = \varnothing$, which indicates that the same intermediate inputs cannot be consumed and produced by the same sub-unit simultaneously.

The relations among the inputs, outputs and intermediate inputs of the DMU_o and its sub-units can be described as follows:

$$x_{io} = \sum_{\substack{k=1 \\ i \in I(k)}}^{K} x_{io}^{k}, \quad i = 1, \ldots, m, \tag{1}$$

$$y_{ro} = \sum_{\substack{k=1 \\ r \in O(k)}}^{K} y_{ro}^{k}, \quad r = 1, \ldots, s, \tag{2}$$

$$z_{do} = \sum_{\substack{k=1 \\ d \in D^{in}(k)}}^{K} z_{do}^{k} = \sum_{\substack{k=1 \\ d \in D^{out}(k)}}^{K} z_{do}^{k}, \quad d = 1, \ldots, D. \tag{3}$$

where x_{io}^{k} is the i-th input for the k-th sub-unit in DMU_o; y_{ro}^{k} is the r-th output of the k-th sub-unit in DMU_o; z_{do}^{k} is the d-th intermediate input of the k-th sub-unit in DMU_o; x_{io} is the i-th input of DMU_o; y_{ro} is the r-th output of DMU_o and z_{do} is the d-th intermediate input of DMU_o. Equation (3) implies that the quantity of the d-th intermediate input produced by the sub-units in DMU_o equals to that consumed by the sub-units in DMU_o. Therefore, the intermediate inputs are fully consumed inside the DMUs.

2.1 The basic models of the network DEA

2.1.1 Series structure

There are K sub-units which are placed alongside each other within each DMU. Each sub-unit produces some intermediate inputs which are then forwarded only to consecutive sub-unit. Such a setting is referred to as a series structure. Figure 1 represents the series structure. Note that the sub-units are not independent in this setting.

2.1.2 Parallel structure

There are K sub-units in each DMU, but they are not inter-connected through the intermediate inputs. Figure 2 represents this structure. It can be observed that each sub-unit only consumes the initial inputs and produces the final outputs. There is no relationship among these sub-units, which means they are independent.

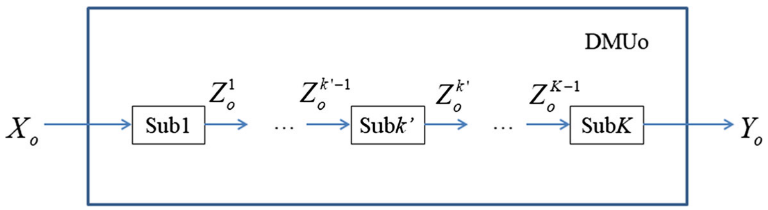

Fig. 1 Series structure

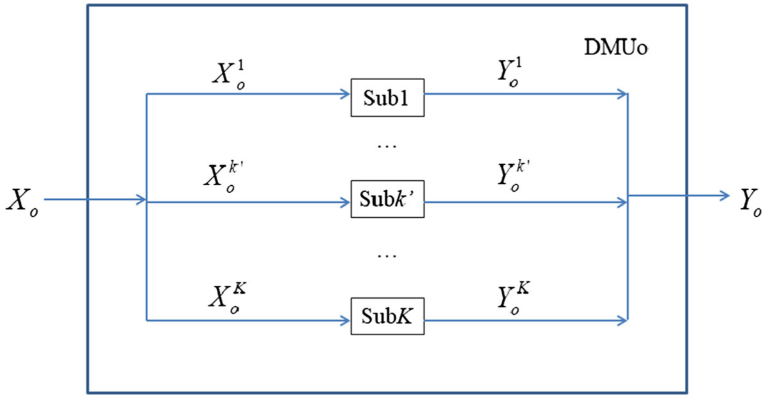

Fig. 2 Parallel structure

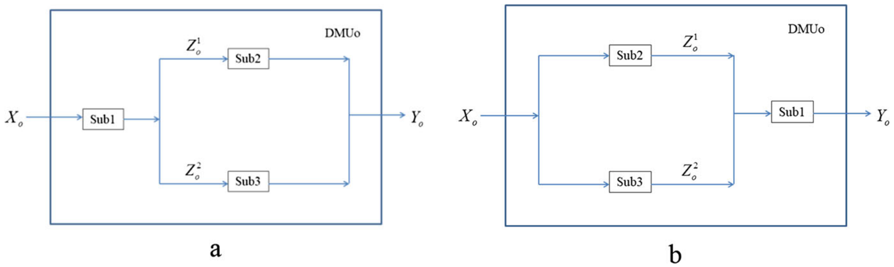

Fig. 3 Mixed structure. **a** One-to-two. **b** Two-to-one

2.1.3 Mixed structure

It is a combination of the series and parallel structures. The mixed structure can be categorized into two kinds, which are represented by Fig. 3a, b. The figures present the simplified models of the mixed structure, as there are only three sub-units. They are referred to as the one-to-two and two-to-one models respectively. The complex mixed structure can be generalized into one-to-many, many-to-one and many-to-many. In the structure, some sub-units produce intermediate inputs for the other sub-units. A sub-unit is dependent on the sub-units that are connected through the flows of the intermediate inputs.

2.2 Sun network structure and the associated static DEA model

For network DEA models with different structures, different mathematical programming problems are applied to evaluate the overall efficiency of the DMU and its sub-unit efficiency. However, there is no model which can be used to evaluate the DMUs with different network structures. In this section, we present the sun structure network DEA model which can represent any structure. Therefore, DEA model based on the sun structure can be used to evaluate the DMUs with different network structure.

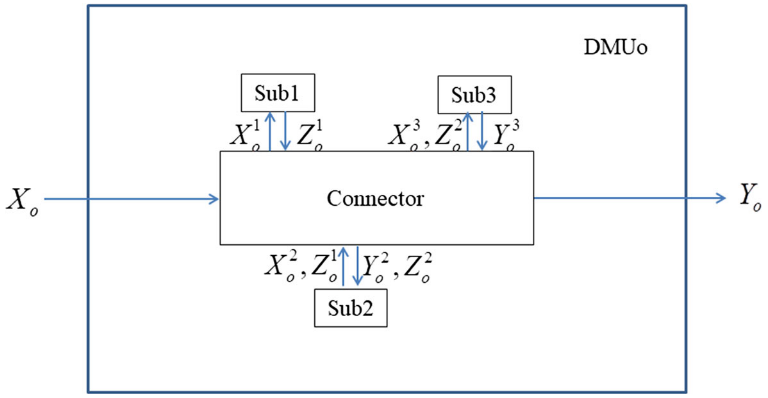

Fig. 4 The static sun network structure

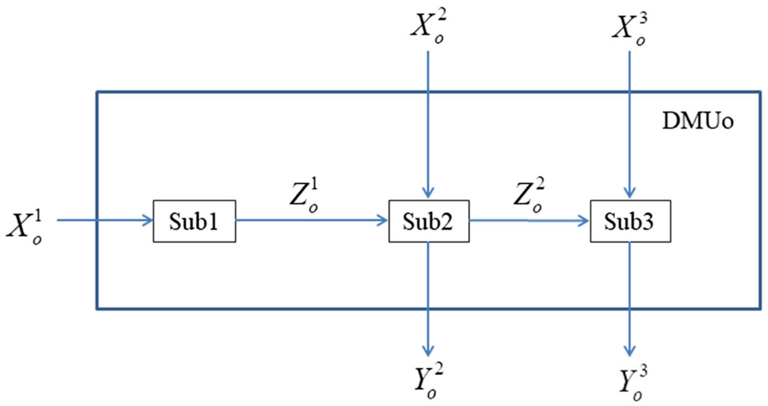

Fig. 5 The old structure

We introduce a dummy connector sub-unit in the sun structure. All the sub-units are then connected to the dummy connector by considering the flows of inputs, outputs and intermediate inputs. All the inputs are allocated to the corresponding sub-units through the dummy connector. If some sub-units are connected through the intermediate inputs, the dummy connector can play the connective role by accepting the intermediate inputs and then allocating them to the corresponding sub-units. The final outputs are also passed over to the dummy connector. Figure 4 depicts the new sun network structure. The new structure represents the old structure as shown in Fig. 5.

Distribution of all the inputs, outputs and intermediate inputs within a DMU is facilitated by the dummy connector. Therefore, the efficiency of a DMU is equal to that of the dummy connector. It can be easily observed that any network structure can be transformed into the sun network structure. Assuming the sun structure, there is no need to pick a specific kind of the network structure before evaluating the efficiency of the DMUs by means of the network DEA. The performance of a sub-unit would not be influenced by other sub-units because the relations among the sub-units are fully facilitated by the dummy connector. By considering the initial inputs, final outputs and intermediate inputs one does not need to impose the same structure for all the DMUs. For example, a DMU with three sub-units following the parallel structure, a DMU with four sub-units following the series structure and a DMU with five sub-units following the mixed structure can all be benchmarked by applying the concept of the sun network structure. As long as the number of inputs, outputs and intermediate inputs are the same across the DMUs, they can be evaluated by the virtue of the dummy connector and the sun network structure. These properties show the superiority of the proposed approach if contrasted to the traditional network DEA models.

Next, we need to construct the efficiency evaluation model for the sun network structure. Initially, the productive technology should be defined. The excessive use of inputs and the shortage in production of outputs represent the inefficiency. The intermediate inputs should also be taken into consideration in the network DEA. The intermediate inputs are used not only as the outputs and but also as the inputs within a DMU. Therefore, they may show both excess and shortage. Assuming constant returns-to-scale and taking into account the considerations above, the productive technology for the sun network structure can be defined as follows:

$$T = \Big\{(X, Y, Z) : \begin{array}{l} X \geq \sum_{j=1}^{n} \lambda_j X_j, \\ Y \leq \sum_{j=1}^{n} \lambda_j Y_j, \\ Z = \sum_{j=1}^{n} \lambda_j Z_j + S^{+z} - S^{-z}, \\ \quad \lambda_j, S^{+z}, S^{-z} \geq 0, \quad j = 1, 2, \ldots, n \end{array} \Big\}, \tag{4}$$

Considering the slack variables for the intermediate inputs, we cannot make sure whether it is negative or positive because they may work as the inputs or outputs for different sub-units. Therefore, we decompose the slack variables into mutually exclusive positive and negative parts. Based on the technology in (4), we can present the model to evaluate the efficiency of a certain DMU_0:

$$\theta_o^* = \text{Min} \frac{1 - \frac{1}{m+D}\left(\sum_{i=1}^{m} \frac{s_{io}^-}{x_{io}} + \sum_{d=1}^{D} \frac{s_{do}^{-z}}{z_{do}}\right)}{1 + \frac{1}{s+D}\left(\sum_{r=1}^{s} \frac{s_{ro}^+}{y_{ro}} + \sum_{d=1}^{D} \frac{s_{do}^{+z}}{z_{do}}\right)},$$

$$\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} - s_{io}^-, i = 1, \ldots, m,$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} = y_{ro} + s_{ro}^+, r = 1, \ldots, s,$$

$$\sum_{j=1}^{n} \lambda_j z_{dj} = z_{do} + s_{do}^{+z} - s_{do}^{-z},$$

$$s_{do}^{+z} \leq Mz_{do}^{+},\ s_{do}^{-z} \leq Mz_{do}^{-},$$
$$z_{do}^{+} + z_{do}^{-} \leq 1,\ z_{do}^{+}, z_{do}^{-} \in \{0, 1\},\ d = 1, 2, \ldots, D,$$
$$\lambda_j, s_{io}^{-}, s_{ro}^{+}, s_{do}^{+z}, s_{do}^{-z} \geq 0. \tag{5}$$

where s_{io}^{-} represents the excess slack variable while s_{ro}^{+} represents the shortage slack variable and M is a large number. If the optimal solution s_d^{+z*} is above zero, the d-th intermediate input is considered as an output of the DMU with a shortfall in producing it in at least one of the sub-units if compared to the other DMUs. Conversely, if the optimal solution s_d^{-z*} is above zero, the d-th intermediate input acts as an input of the DMU with a surplus in it in at least one of the sub-units if compared to the other DMUs. Because of the fourth and fifth constraints, s_d^{+z*} and s_d^{-z*} cannot be positive and negative simultaneously, which indicates that the intermediate input cannot work as both the input and the output of a DMU.

Remark 1 To avoid vagueness on the possible dual roles of intermediates, the model (5) can be transformed into the following forms. Model (5a) and model (5b) indicate the cases when intermediates work as inputs and outputs of a DMU respectively:

$$\theta_o^* = \text{Min} \frac{1 - \frac{1}{m+D}\left(\sum_{i=1}^{m} \frac{s_{io}^{-}}{x_{io}} + \sum_{d=1}^{D} \frac{s_{do}^{-z}}{z_{do}}\right)}{1 + \frac{1}{s}\left(\sum_{r=1}^{s} \frac{s_{ro}^{+}}{y_{ro}}\right)}$$
$$s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} - s_{io}^{-}, \quad i = 1, \ldots, m,$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} = y_{ro} + s_{ro}^{+}, \quad r = 1, \ldots, s,$$
$$\sum_{j=1}^{n} \lambda_j z_{dj} = z_{do} - s_{do}^{-z},$$
$$s_{do}^{-z} \leq M, \quad \lambda_j, s_{io}^{-}, s_{ro}^{+}, s_{do}^{-z} \geq 0. \tag{5a}$$

$$\theta_o^* = \text{Min} \frac{1 - \frac{1}{m}\left(\sum_{i=1}^{m} \frac{s_{io}^{-}}{x_{io}}\right)}{1 + \frac{1}{s+D}\left(\sum_{r=1}^{s} \frac{s_{ro}^{+}}{y_{ro}} + \sum_{d=1}^{D} \frac{s_{do}^{+z}}{z_{do}}\right)}$$
$$s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} - s_{io}^{-}, \quad i = 1, \ldots, m,$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} = y_{ro} + s_{ro}^{+}, \quad r = 1, \ldots, s,$$

$$\sum_{j=1}^{n} \lambda_j z_{dj} = z_{do} + s_{do}^{+z},$$
$$s_{do}^{+z} \le M,\ \lambda_j, s_{io}^{-}, s_{ro}^{+}, s_{do}^{+z} \ge 0. \tag{5b}$$

We do not know which intermediate inputs are treated as the initial inputs or final outputs prior to running model (5). Therefore, we have considered all the intermediate inputs in the objective function of (5) when evaluating the efficiency. In this regard, the efficiency scores based on (5) can be termed as the pseudo efficiency scores. After solving the above optimization programming, we, indeed, obtain the information on which intermediate inputs act as the initial inputs or final outputs. Based on this, we can redefine the efficiency score. Let the variables with the asterisk represent the optimal solution of (5). Then, the efficiency of $\mathrm{DMU_O}$ is redefined *ex post* as follows:

$$\theta_o^{\prime *} = \frac{1 - \frac{1}{m+D(I)}\left(\sum_{i=1}^{m} \frac{s_{io}^{*-}}{x_{io}} + \sum_{d=1}^{D(I)} \frac{s_{do}^{*-z}}{z_{do}}\right)}{1 + \frac{1}{s+D(O)}\left(\sum_{r=1}^{s} \frac{s_{ro}^{*+}}{y_{ro}} + \sum_{d=1}^{D(O)} \frac{s_{do}^{*+z}}{z_{do}}\right)}, \tag{6}$$

where $D(I)$ represents the number of the intermediate inputs which act as the initial inputs and $D(O)$ represents the number of the intermediate inputs which act as the final outputs. The refined efficiency score is more consistent with the objective function of the SBM model and better reflects the performance of $\mathrm{DMU_o}$. We also notice that $\theta_o^{\prime *} \le \theta_o^*$, which indicates the higher discriminating power of the refined measure. Therefore, we term the refined efficiency as the true efficiency of $\mathrm{DMU_o}$.

Definition 1 An $\mathrm{DMU_o}$ is called an efficient if and only if θ_o^* and/or $\theta_o^{\prime *}$ equals unity.

Theorem 1 *For an efficient DMU, the efficiency score is always unity, no matter whether* (5) *or* (6) *is applied.*

Proof The proof is obvious. Models (5a) and (5b) are slacks-based measure (SBM) DEA proposed by Tone (2001), which demonstrated the relationship between CCR-efficiency and SBM-efficiency and asserted that a DMU is SBM-efficient if and only if it is CCR-efficient. Associated with CCR model, at least one DMU's CCR-efficiency is unity (see e.g. Seiford and Zhu 1999b). Thus, we can easily see that Models (5a) and (5b) have at least one DMU whose efficiency score is unity. See Tone (2001). □

So far, we have defined the efficiency of $\mathrm{DMU_o}$. Now, we turn to definition of define the efficiency of a sub-unit within $\mathrm{DMU_o}$. We assume $\left(\lambda_j^*, s_i^{-*}, s_r^{+*}, s_d^{+z*}, s_d^{-z*}\right)$ is the optimal solution of (5). We have to obtain the slack variables for a sub-unit within the DMU to define the efficiency of the sub-unit. Here, we take the k-th sub-unit in the DMU as an example:

$$s_i^{-k*} = x_{io}^k - \sum_{j=1}^{n} \lambda_j^* x_{ij},\ i \in I(k), \tag{7}$$

$$s_r^{+k*} = \sum_{j=1}^{n} \lambda_j^* y_{rj} - y_{ro}^k, \quad r \in O(h), \tag{8}$$

$$s_d^{+zh*} - s_d^{-zh*} = \sum_{j=1}^{n} \lambda_j^* z_{dj} - z_{do}, d \in D^{out}(h) \cup D^{in}(h), \tag{9}$$

We have assumed that $D^{out}(h) \cap D^{in}(h) = \varnothing$, which indicates that any intermediate input cannot be consumed and produced by the same sub-unit simultaneously. If d is the element of $D^{out}(h)$, s_d^{-zh*} must be zero. If d is the element of $D^{in}(h)$, s_d^{+zh*} must be zero.

Before defining the efficiency of each sub-unit, we also need to take into consideration the relations among the slack variables for the DMU and its sub-units:

$$s_i^{-*} = \sum_{\substack{k=1 \\ i \in I(k)}}^{K} s_i^{-k*}, \quad i = 1, \ldots, m, \tag{10}$$

$$s_r^{+*} = \sum_{\substack{k=1 \\ r \in O(k)}}^{K} s_r^{+k*}, \quad r = 1, \ldots, s, \tag{11}$$

$$s_d^{+z*} - s_d^{-z*} = 2\left(\sum_{\substack{k=1 \\ d \in D^{out}(k)}}^{K} s_d^{+zk*} - \sum_{\substack{k=1 \\ d \in D^{in}(k)}}^{K} s_d^{-zk*} \right), \quad d = 1, \ldots .D. \tag{12}$$

Note that the intermediate inputs work as either the initial inputs or the final outputs from the system point of view. However, looking inside each DMU, the intermediate inputs act as both the inputs and outputs there. Therefore, the sum of the slack variables of the intermediate inputs for the DMU is twice the sum of the slack variables of the intermediate inputs for the sub-units within the DMU.

Now we can get the efficiency of the kth sub-unit in the DMU evaluated, namely $\mathrm{DMU_o}$, based on the relations among the slack variables of the DMU and its sub-units:

$$\theta_o^{k*} = \frac{1 - \frac{1}{\widehat{i}(k) + \widehat{d}^{in}(k)} \left(\sum_{i \in I(k)} \frac{s_i^{-k*}}{x_{io}^k} + \sum_{d \in D^{in}(k)} \frac{s_d^{-zk*}}{z_{do}^k} \right)}{1 + \frac{1}{\widehat{o}(k) + \widehat{d}^{out}(k)} \left(\sum_{r \in O(k)} \frac{s_r^{+k*}}{y_{ro}^k} + \sum_{d \in D^{out}(k)} \frac{s_d^{+zk*}}{z_{do}^k} \right)}, \tag{13}$$

where $\overset{\frown}{d}^{in}(k)$ refers to the number of intermediate inputs which are consumed by the k-th sub-unit, $\overset{\frown}{d}^{out}(k)$ refers to the number of intermediate inputs which are produced by the k-th sub-unit, $\widehat{i}(k)$ refers to the number of inputs of the k-th sub-unit, and $\widehat{o}(k)$ refers to the number of outputs of the k-th sub-unit.

Definition 2 The k-th sub-unit of the DMU evaluated is efficient one if and only if θ_o^{k*} equals to unity.

Theorem 2 *DMU_o is efficient if and only if each sub-unit in the DMU is efficient.*

Proof We first prove if each sub-unit in the DMU is efficient, DMU_o is efficient. We take the k-th sub-unit as an example. If the k-th sub-unit is efficient, θ_o^{k*} equals to unity. All the slack variables in (13) should be zero, otherwise θ_o^{k*} would be smaller than unity. It can be generalized to other sub-units of the DMU. Finally, we can get all the slack variables of each sub-unit in the DMU are zero. According to relations among the slack variables of the DMU and its sub-units, we can get that all the slack variables of the DMU are zero. Therefore, θ_o^* equals to unity. The DMU is efficient.

If DMU_o is efficient, each sub-unit in the DMU is also efficient. It can be proved in a similar way. □

Corollary 1 *DMU_o is inefficient if and only if at least one of the sub-units is inefficient.*

Note that the models presented here are the non-oriented ones. As for the input-oriented and output-oriented cases, one can obtain them by modifying the non-oriented case.

Environmentally sensitive measures of the efficiency require incorporating the undesirable outputs into analysis. Indeed, these are the inevitable part of much production processes. Here, we will shortly discuss the issue of the sun network structure DEA with undesirable outputs. First, we construct the technology including the undesirable output. We follow a simplistic approach relying on the assumption of strong disposability of the undesirable outputs. Essentially, this approach treats the undesirable outputs as the inputs:

$$
\begin{aligned}
T = \{(X, Y, U, Z):\ & X \geq \sum_{j=1}^{n} \lambda_j X_j, \\
& Y \leq \sum_{j=1}^{n} \lambda_j Y_j, \\
& U \geq \sum_{j=1}^{n} \lambda_j U_j, \\
& Z = \sum_{j=1}^{n} \lambda_j Z_j + S^{+z} - S^{-z}, \\
& \qquad \lambda_j, S^{+z}, S^{-z} \geq 0,\ j = 1, 2, \ldots, n\Big\},
\end{aligned}
\tag{14}
$$

where U_j is the vector of the undesirable outputs for the j-th DMU.

Based on the technology in (14), we can construct the DEA model considering the undesirable outputs for the sun network structure:

$$
\begin{aligned}
&\theta_o^* = \text{Min} \frac{1 - \frac{1}{m+D+L}\left(\sum_{i=1}^{m} \frac{s_{io}^-}{x_{io}} + \sum_{d=1}^{D} \frac{s_{do}^{-z}}{z_{do}} + \sum_{l=1}^{L} \frac{s_{lo}^{-l}}{u_{lo}}\right)}{1 + \frac{1}{s+D}\left(\sum_{r=1}^{s} \frac{s_{ro}^+}{y_{ro}} + \sum_{d=1}^{D} \frac{s_{do}^{+z}}{z_{do}}\right)}, \\
&\text{s.t.} \\
&\sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} - s_{io}^-, i = 1, \ldots, m, \\
&\sum_{j=1}^{n} \lambda_j y_{rj} = y_{ro} + s_{ro}^+, r = 1, \ldots, s, \\
&\sum_{j=1}^{n} \lambda_j u_{lj} = u_{lo} - s_{lo}^{-u}, l = 1, \ldots, L, \\
&\sum_{j=1}^{n} \lambda_j z_{dj} = z_{do} + s_{do}^{+z} - s_{do}^{-z}, \\
&s_{do}^{+z} \le M z_{do}^+,\ s_{do}^{-z} \le M z_{do}^-, \\
&z_{do}^+ + z_{do}^- \le 1,\ z_{do}^+, z_{do}^- \in \{0, 1\},\ d = 1, 2, \ldots, D, \\
&\lambda_j, s_{io}^-, s_{ro}^+, s_{do}^{+z}, s_{do}^{-z} \ge 0.
\end{aligned} \tag{15}
$$

We can also get the efficiency of each sub-unit considering the undesirable outputs following (13).

2.3 Dynamic DEA model based on the sun structure

In this sub-section, we construct a dynamic DEA model based on the sun network structure. We suppose there are T time periods indexed over t. The following notations are used in the dynamic setting:

Input–output data

- x_{iot}^k Input i for the k-th sub-unit of DMU_o in period t;
- y_{rot}^k Output r from the k-th sub-unit of DMU_o for in period t;
- z_{dot}^k Intermediate input d for the k-th sub-unit of DMU_o in period t;
- p_{cot}^k Carry-over product c produced by the k-th sub-unit of DMU_o in period t;

Decision variables

- s_{iot}^{-k} Slack variable for input i for the k-th sub-unit of DMU_o in period t;
- s_{rot}^{+k} Slack variable for output r for the k-th sub-unit of DMU_o in period t;
- s_{dot}^{-zk} Excess slack variable for intermediate input d for the k-th sub-unit of DMU_o in period t;

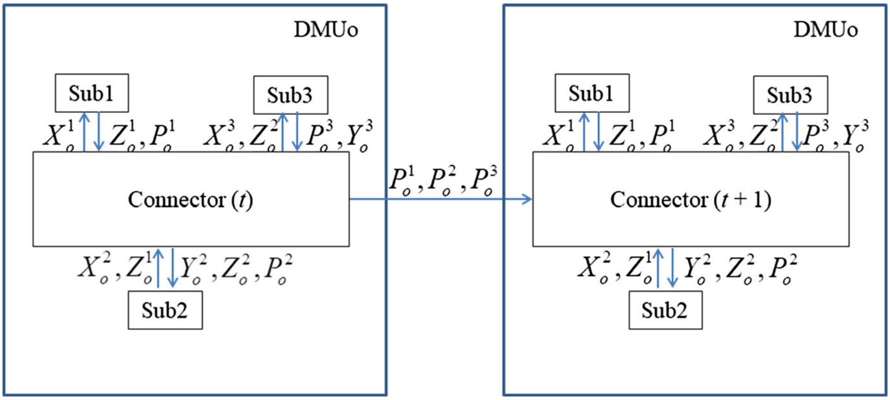

Fig. 6 The dynamic sun network structure

s_{dot}^{+zk} Shortage slack variable for intermediate input d for the k-th sub-unit of DMU_o in period t;

s_{cot}^{-ck} Excess slack variable for carry-over product c for the k-th sub-unit of DMU_o in period t;

s_{cot}^{+ck} Shortage slack variable for carry-over product c for the k-th sub-unit of DMU_o in period t;

λ_j^t Intensity variable for DMU_j in period t.

We introduce a dummy connector sub-unit in each period the structure operates in. In each period, all the inputs, outputs and intermediate inputs are distributed through the dummy connector. Different periods are connected through the carry-over factors (products). The dummy connectors also allocate the carry-over factors across different time periods. Figure 6 shows the dynamic setting the sun network structure. The old structure is presented in Fig. 7. In each period, the carry-over products act as a part of the final outputs for that period and as a part of the initial inputs for the next period. As for the relationship between the carry-over products of the DMU and its sub-units, we have $p_{\text{cot}} = \sum_{\substack{k=1 \\ c \in P^{out}(t,k)}}^{K} p_{\text{cot}}^k$, $c = 1, \ldots, C, t = 1, \ldots, T$, where p_{cot} refers to the c-th carry-over product produced by DMU_o in period t and p_{cot}^k refers to the c-th carry-over product produced by the k-th sub-unit of DMU_o in period t; $P^{out}(t, k)$ denotes the set of the carry-over products produced by the k-th sub-unit in period t. We still assume that all the carry-over products of a certain period are used for the next period.

We can establish the dynamic DEA model for the sun network structure as follows:

$$\theta_o^{**} = \text{Min} \frac{\left\{ \begin{array}{l} w^1 \left[1 - \frac{1}{m+D} \left(\sum_{i=1}^{m} \frac{s_{iot}^-}{x_{iot}} + \sum_{d=1}^{D} \frac{s_{dot}^{-z}}{s_{dot}^{-z}} \right) \right] \\ + \sum_{t=2}^{T} w^t \left[1 - \frac{1}{m+D+C} \left(\sum_{i=1}^{m} \frac{s_{iot}^-}{x_{iot}} + \sum_{d=1}^{D} \frac{s_{dot}^{-z}}{s_{dot}^{-z}} + \sum_{c=1}^{C} \frac{s_{cot}^{-c}}{p_{cot-1}} \right) \right] \end{array} \right\}}{\sum_{t=1}^{T} w^t \left[1 + \frac{1}{s+D+C} \left(\sum_{r=1}^{s} \frac{s_{rot}^+}{y_{rot}} + \sum_{d=1}^{D} \frac{s_{dot}^{+z}}{z_{dot}} + \sum_{c=1}^{C} \frac{s_{cot}^{+c}}{p_{cot}} \right) \right]}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j^t x_{ijt} = x_{iot} - s_{iot}^{-}, \quad t = 1, \ldots, T, i = 1, \ldots, m,$$

$$\sum_{j=1}^{n} \lambda_j^t y_{rjt} = y_{rot} + s_{rot}^{+}, \quad t = 1, \ldots, T, r = 1, \ldots, s,$$

$$\sum_{j=1}^{n} \lambda_j^t z_{djt} = z_{dot} + s_{dot}^{+z} - s_{dot}^{-z}, \quad t = 1, \ldots, T, d = 1, \ldots, D,$$

$$s_{dot}^{+z} \leq M z_{dot}^{+}, \; s_{dot}^{-z} \leq M z_{dot}^{-},$$

$$z_{dot}^{+} + z_{dot}^{-} \leq 1, \; z_{dot}^{+}, z_{dot}^{-} \in \{0, 1\}, \quad t = 1, \ldots, T, d = 1, \ldots, D,$$

$$\sum_{j=1}^{n} \lambda_j^t p_{cjt} = p_{\text{cot}} + s_{\text{cot}}^{+c}, \quad t = 1, \ldots, T, c = 1, \ldots, C,$$

$$\sum_{j=1}^{n} \lambda_j^t p_{cjt-1} = p_{\text{cot}-1} - s_{\text{cot}}^{-c}, t = 2, \ldots, T, c = 1, \ldots, C,$$

$$\lambda_j^t, s_{iot}^{-}, s_{rot}^{+}, s_{dot}^{-z}, s_{dot}^{+z}, s_{\text{cot}}^{-c}, s_{\text{cot}}^{+c} \geq 0. \tag{16}$$

The constraint for the weights associated with time periods is $\sum_{t=1}^{T} w^t = 1$. The weights for different periods should be determined in advance. In Model (16), all the intermediate inputs are allocated inside the DMU of each period and they either work as the inputs or the outputs for this period and DMU. As for the carry-over products, they are allocated through different connectors. The connector allocates the carry-over products to the next connector. They work as both the inputs for the current period and the outputs for the preceding period. Overall, there is a two-way relationship between the sub-units and the connector in the DMU for a certain time period while there is the one-way relationship between the adjacent connectors, which can be shown in Fig. 6.

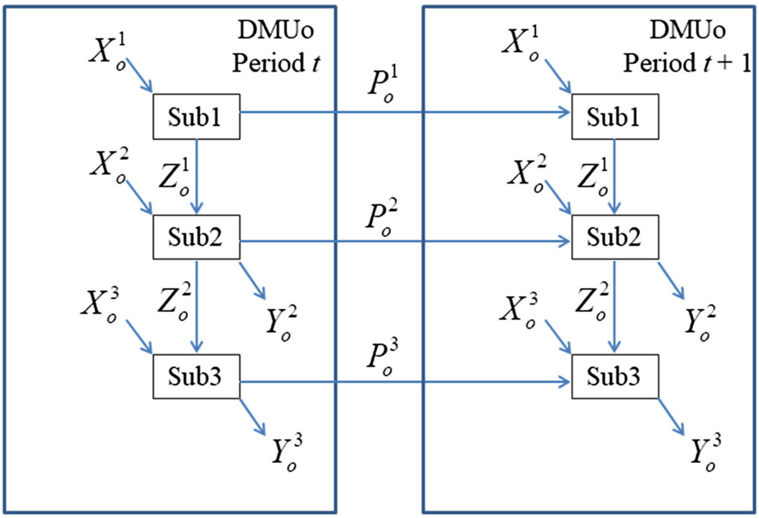

Fig. 7 The old network structure in the dynamic setting

The efficiency obtained via (16) is also the pseudo efficiency of DMU_o. In order to get the true efficiency of DMU_o, we refine the efficiency *ex post* as follows:

$$\theta_o^{**\prime} = \frac{\left\{ \begin{array}{l} w^1\left[1 - \frac{1}{m + D'(1)}\left(\sum_{i=1}^{m} \frac{s_{io1}^{-*}}{x_{io1}} + \sum_{d=1}^{D\prime(1)} \frac{s_{do1}^{-z*}}{z_{do1}}\right)\right] \\ + \sum_{t=2}^{T} w^t\left[1 - \frac{1}{m + D'(t) + C}\left(\sum_{i=1}^{m} \frac{s_{iot}^{-*}}{x_{iot}} + \sum_{d=1}^{D\prime(t)} \frac{s_{dot}^{-z*}}{z_{dot}} + \sum_{c=1}^{C} \frac{s_{cot}^{-c*}}{p_{cot-1}}\right)\right] \end{array} \right\}}{\sum_{t=1}^{T} w^t\left[1 + \frac{1}{S+D''(t)+C}\left(\sum_{r=1}^{s} \frac{s_{rot}^{+*}}{y_{rot}} + \sum_{d=1}^{D''(t)} \frac{s_{dot}^{+z*}}{z_{dot}} + \sum_{c=1}^{C} \frac{s_{cot}^{+c*}}{p_{cot}}\right)\right]} \tag{17}$$

where variables with an asterisk represent the optimal solution of (16), $D'(t)$ refers to the number of the intermediate inputs which act as the inputs in period t, and $D''(t)$ refers to the number of the intermediate inputs which act as the outputs in period t. The redefined efficiency measure is more consistent with the objective function of the SBM model and better reflects the performance of DMU_o. Therefore, we refer to this measure as the true efficiency of DMU_o.

Theorem 3 *As for the efficient DMUs, the true and pseudo efficiency are the same. It is obvious, as a result, we omit its proof.*

Definition 3 If θ_o^{**} or $\theta_o^{**'}$ equals to unity, DMU_o is overall efficient. In this case, $s_{iot}^{-}=0\ (\forall i, t)$, $s_{rot}^{+} = 0\ (\forall r, t)$, $s_{dot}^{-z} = s_{dot}^{+z} = 0\ (\forall d, t)$ and $s_{cot}^{-c}=s_{cot}^{+c}=0\ (\forall c, t)$.

We can also define the efficiency of each sub-unit, each period and each sub-unit in each period. We assume s_{ijt}^{-*}, s_{rjt}^{+*}, s_{djt}^{-z*}, s_{djt}^{+z*}, s_{cjt}^{+c*} and s_{cjt}^{-c*} are the optimal solutions of (16). The efficiency of each period can be defined as follows:

$$\theta_{T_1}^{**} = \sum_{j=1}^{n} \varpi_j \left[\frac{1 - \frac{1}{m+D_j'(1)}\left(\sum_{i=1}^{m} \frac{s_{ij1}^{-*}}{x_{ij1}} + \sum_{d=1}^{D_j'(1)} \frac{s_{dj1}^{-z*}}{z_{dj1}}\right)}{1 + \frac{1}{S+D_j''(1)+C}\left(\sum_{r=1}^{s} \frac{s_{rj1}^{+*}}{y_{rj1}} + \sum_{d=1}^{D_j''(1)} \frac{s_{dj1}^{+z*}}{z_{dj1}} + \sum_{c=1}^{C} \frac{s_{cj1}^{+c*}}{p_{cj1}}\right)} \right], \tag{18}$$

$$\theta_{T_t}^{**} = \sum_{j=1}^{n} \varpi_j \left[\frac{1 - \frac{1}{m+D_j'(t)+C}\left(\sum_{i=1}^{m} \frac{s_{ijt}^{-*}}{x_{ijt}} + \sum_{d=1}^{D_j'(t)} \frac{s_{djt}^{-z*}}{z_{djt}} + \sum_{c=1}^{C} \frac{s_{cjt}^{-c*}}{p_{cjt-1}}\right)}{1 + \frac{1}{S+D_j''(t)+C}\left(\sum_{r=1}^{s} \frac{s_{rjt}^{+*}}{y_{rjt}} + \sum_{d=1}^{D_j''(t)} \frac{s_{djt}^{+z*}}{z_{djt}} + \sum_{c=1}^{C} \frac{s_{cjt}^{+c*}}{p_{cjt}}\right)} \right],$$
$$t = 2, \ldots, T, \tag{19}$$

where ϖ_j represents the importance of each DMU. The bigger the ϖ_j is, the more important DMU_j is. The constraint for ϖ_j is $\sum_{j=1}^{n} \varpi_j = 1$. $D_j'(t)$ refers to the number of the intermediate inputs in DMU_j which act as the inputs in period t, and $D_j''(t)$ refers to the number of the intermediate inputs in DMU_j which act as the outputs in period t. We apply different measures of efficiency for the first period and the rest ones because the first period receives no carry-over products, whereas the other periods receive these from the preceding periods. Accordingly, excess slack variable associated with of the carry-over products is absent from the numerator of (18).

Before defining the efficiency of each sub-unit, we also need to consider the slack variables of the sub-units in the DMU. They can be obtained as follows:

$$s_{iot}^{-k*} = x_{iot}^{k} - \sum_{j=1}^{n} \lambda_j^t x_{ij}, i \in I(k), \tag{20}$$

$$s_{rot}^{+k*} = \sum_{j=1}^{n} \lambda_j^t y_{rj} - y_{rot}^{k}, r \in O(k), \tag{21}$$

$$s_{dot}^{+zk*} - s_{dot}^{-zk*} = \sum_{j=1}^{n} \lambda_j^t z_{dj} - z_{dot}, \quad d \in D^{out}(k) \cup D^{in}(k), \tag{22}$$

$$s_{\mathrm{cot}}^{-ck} = p_{\mathrm{cot}-1} - \sum_{j=1}^{n} \lambda_j^t p_{cjt-1}, \quad c \in P^{out}(t-1, k), \tag{23}$$

$$s_{\mathrm{cot}}^{+ck} = \sum_{j=1}^{n} \lambda_j^t p_{cjt} - p_{\mathrm{cot}}, \quad c \in P^{out}(t, k). \tag{24}$$

Note that when d is the element of $D^{out}(k)$, s_{dot}^{-zk*} should be zero and when d is the element of $D^{in}(k)$, s_{dot}^{+zk*} should be zero. $P^{out}(t-1, k)$ denotes the set of the carry-over products which are produced by the k-th sub-unit in period $t-1$. $P^{out}(t, k)$ denotes the set the carry-over products which are produced by the k-th sub-unit in period t.

Therefore, the efficiency of each sub-unit based on the dynamic DEA model of the sun structure can be defined as follows:

$$\theta_k^{**} = \frac{\left\{ \begin{array}{l} w^1 \left[1 - \frac{1}{\widehat{i}(k) + \widehat{d}^{in}(k)} \left(\sum_{i \in I(k)} \frac{s_{io1}^{-k*}}{x_{io1}^{k}} + \sum_{d \in D^{in}(k)} \frac{s_{do1}^{-zk*}}{z_{do1}^{k}} \right) \right] \\ + \sum_{t=2}^{T} w^t \left[1 - \frac{1}{\widehat{i}(k) + \widehat{d}^{in}(k) + \widehat{p}^{out}(t-1, k)} \left(\sum_{i \in I(k)} \frac{s_{iot}^{-k*}}{x_{iot}^{k}} + \sum_{d \in D^{in}(k)} \frac{s_{dot}^{-zk*}}{z_{dot}^{k}} + \sum_{c \in P^{out}(t-1.k)} \frac{s_{cot}^{-ck*}}{p_{\mathrm{cot}\ -\ 1}^{k}} \right) \right] \end{array} \right\}}{\sum_{t=1}^{T} w^t \left[1 + \frac{1}{\widehat{o}(k) + \widehat{d}^{out}(k) + \widehat{p}^{out}(t,k)} \left(\sum_{r \in O(k)} \frac{s_{rot}^{+k*}}{y_{rot}^{k}} + \sum_{d \in D^{out}(k)} \frac{s_{dot}^{+zk*}}{z_{dot}^{k}} + \sum_{c \in P^{out}(t,k)} \frac{s_{cot}^{+ck*}}{p_{\mathrm{cot}}^{k}} \right) \right]}, \tag{25}$$

where $\widehat{p}^{out}(t, k)$ refers to the number of carry-over products which are produced by the k-th sub-unit of $\mathrm{DMU_o}$ in period t.

Finally, the efficiency of each sub-unit in each period can be defined as:

$$\theta_k^{t**}=\frac{1-\frac{1}{\widehat{i}(k)+\widehat{d}^{in}(k)+\widehat{p}^{out}(t-1,k)}\left(\sum_{i\in I(k)}\frac{s_{iot}^{-k*}}{x_{iot}^k}+\sum_{d\in D^{in}(k)}\frac{s_{dot}^{-zk*}}{z_{dot}^k}+\sum_{c\in P^{out}(t-1,k)}\frac{s_{cot}^{-ck*}}{p_{\cot-1}^k}\right)}{1+\frac{1}{\widehat{o}(k)+\widehat{d}^{out}(k)+\widehat{p}^{out}(t,k)}\left(\sum_{r\in O(k)}\frac{s_{rot}^{+k*}}{y_{rot}^k}+\sum_{d\in D^{out}(k)}\frac{s_{dot}^{+zk*}}{z_{dot}^k}+\sum_{c\in P^{out}(t,k)}\frac{s_{cot}^{+ck*}}{p_{\cot}^k}\right)},$$
$$t=2,\ldots,T,k=1,\ldots,K, \tag{26}$$

$$\theta_k^{1**}=\frac{1-\frac{1}{\widehat{i}(k)+\widehat{d}^{in}(k)}\left(\sum_{i\in I(k)}\frac{s_{io1}^{-k*}}{x_{io1}^k}+\sum_{d\in D^{in}(k)}\frac{s_{do1}^{-zk*}}{z_{do1}^k}\right)}{1+\frac{1}{\widehat{o}(k)+\widehat{d}^{out}(k)+\widehat{p}^{out}(t,k)}\left(\sum_{r\in O(k)}\frac{s_{rot}^{+k*}}{y_{rot}^k}+\sum_{d\in D^{out}(k)}\frac{s_{dot}^{+zk*}}{z_{dot}^k}+\sum_{c\in P^{out}(t,k)}\frac{s_{cot}^{+ck*}}{p_{\cot}^k}\right)},$$
$$k=1,\ldots,K. \tag{27}$$

Note that even though the overall efficiency is uniquely determined, the efficiencies for each period, each sub-unit and each sub-unit in each period are not necessarily unique.

The following two theorems deal with the relationship among the efficiency of each sub-unit and each sub-unit of each period.

Theorem 4 *DMU_o is efficient if and only if each sub-unit in this DMU is efficient.*

Theorem 5 *DMU_o is efficient if and only if each sub-unit of each period in this DMU is efficient.*

As for the proof for the above theories, readers can follow the proof of Theorem 2 by the way of analogy.

In this paper, we focus on the static and dynamic network DEA models based on the sun network structure under the constant returns-to-scale. If constraint $\sum_{j=1}^{n}\lambda_j^t=1$ is added, it would turn into the model under the case of variable returns-to-scale (when no undesirable output is involved).

3 Application

In this section, we give some numerical examples to show the operationality of the proposed models. Specifically, we focus on the static and dynamic network DEA models based on the sun network structure.

3.1 The example for the static DEA model based on the sun network structure

The static sun network structure DEA relies on the principles laid out in Sect. 2.2. Let us assume there are three sub-units in each DMU. The first sub-unit produces the intermediate inputs for the second sub-unit. The second sub-unit produces the intermediate inputs for the third sub-unit. The underlying network structure can be shown clearly in Fig. 5. It can be transformed into the sun network structure, which

Table 1 Input-output data for the static network DEA model

DMU	x_1	x_2	x_3	y_1	y_2	z_1	z_2
1	0.838	0.277	0.962	0.879	0.337	0.894	0.362
2	1.233	0.132	0.443	0.538	0.180	0.678	0.188
3	0.321	0.042	0.482	0.911	0.198	0.836	0.207
4	1.483	0.111	0.467	0.570	0.491	0.869	0.516
5	1.592	0.208	1.073	1.086	0.372	0.693	0.407
6	0.790	0.139	0.545	0.722	0.253	0.966	0.269
7	0.451	0.075	0.366	0.509	0.241	0.647	0.257
8	0.408	0.074	0.229	0.619	0.097	0.756	0.103
9	1.864	0.061	0.691	1.023	0.380	1.191	0.402
10	1.222	0.149	0.337	0.769	0.178	0.792	0.187

Table 2 Efficiency scores for the static network DEA example

DMU	SBM	Pseudo efficiency based on (5)	True efficiency based on (6)
1	0.425	0.521	0.441
2	0.400	0.540	0.519
3	1.000	1.000	1.000
4	1.000	1.000	1.000
5	0.415	0.440	0.357
6	0.558	0.723	0.665
7	1.000	1.000	1.000
8	1.000	1.000	1.000
9	1.000	1.000	1.000
10	0.694	0.757	0.672

is depicted in Fig. 4. It can be seen that the dummy connector sub-unit plays the key role in connecting all the sub-units. All the sub-units are independent owing to the introduction of the dummy connector. The input–output data for the empirical example are presented in Table 1.

In this example, $X_o = (x_{1o}, x_{2o}, x_{3o})$, $Y_o = (y_{1o}, y_{2o})$ and $Z_o = (z_{1o}, z_{2o})$ are the initial input, final output and intermediate input vectors for $\mathrm{DMU_o}$ respectively. We then apply (5) and (6) along with the traditional slack-based model (SBM) to evaluate efficiency of each DMU. The resulting efficiency scores are presented in Table 2.

The SBM ignores the network structure in evaluation which is not the case for (5). That is the main reason why the efficiencies differ across the two models. DMUs 3, 4, 7, 8 and 9 are efficient using either the SBM or network DEA as defined by (5). However, one cannot assume the DMUs are always efficient no if SBM or network DEA in (5) is applied. There might exists the case when a DMU is efficient in the network structure evaluated by (5), whereas the SBM shows inefficiency as the inner

Table 3 Sub-unit efficiency scores

DMU	Sub-unit 1	Sub-unit 2	Sub-unit 3
1	0.410	0.676	0.913
2	0.831	0.558	0.716
3	1.000	1.000	1.000
4	1.000	1.000	1.000
5	0.167	1.000	0.900
6	0.613	0.502	0.992
7	1.000	1.000	1.000
8	1.000	1.000	1.000
9	1.000	1.000	1.000
10	0.488	0.920	0.942

Table 4 Data for the dynamic network DEA in Period 1

DMU	x_1	x_2	x_3	y_1	y_2	z_1	z_2	p_1	p_2
1	0.838	0.277	0.962	0.879	0.337	0.894	0.362	0.621	0.876
2	1.233	0.132	0.443	0.538	0.180	0.678	0.188	0.762	0.927
3	0.321	0.042	0.482	0.911	0.198	0.836	0.207	1.027	0.865
4	1.483	0.111	0.467	0.570	0.491	0.869	0.516	0.962	0.765

network structure remains ignored in the latter case. It follows from results in Table 2 that DMUs 3, 4, 7, 8 and 9 are all efficient irrespectively if models (5) or (6) are applied. Indeed, Theorem 3 already stated that the efficient DMUs possess equal true and pseudo efficiency scores. As for the inefficient DMUs, the efficiency scores based on (6) are lower than those based on (5). One can also obtain the efficiency of each sub-unit based on (13). The efficiency scores for each sub-unit are given in Table 3.

As it was already shown in Table 2, DMUS 3, 4, 7, 8 and 9 are fully efficient. Therefore, Table 3 suggests the efficiency of each sub-unit in the five efficient DMUs equals unity. As for the inefficient DMUs, their overall efficiency scores in Table 2 fall within the range bounded by the highest and lowest efficiency of its sub-units given in Table 3.

3.2 The example for the dynamic DEA model based on the sun network structure

The principles of the dynamic sun network structure DEA were presented in Sect. 2.3. In this sub-section, we employ the dynamic DEA model based on the sun network structure to evaluate the DMUs operating in multiple periods. The data for the three subsequent different periods are given in Tables 4, 5 and 6.

Note that the four DMUs are nonhomogeneous in their inner structure. The different structures are shown in Fig. 8. The conventional dynamic network DEA model cannot handle evaluation of their performance. However, the number of inputs, outputs, intermediate inputs and carry-over products are the same for the four DMUs.

Table 5 Data for the dynamic network DEA in Period 2

DMU	x_1	x_2	x_3	y_1	y_2	z_1	z_2	p_1	p_2
1	1.592	0.208	1.073	1.086	0.372	0.693	0.407	0.964	0.583
2	0.790	0.139	0.545	0.722	0.253	0.966	0.269	1.021	0.936
3	0.451	0.075	0.366	0.509	0.241	0.647	0.257	0.763	0.732
4	0.408	0.074	0.229	0.619	0.097	0.756	0.103	0.632	0.628

Table 6 Data for the dynamic network DEA in Period 3

DMU	x_1	x_2	x_3	y_1	y_2	z_1	z_2	p_1	p_2
1	1.864	0.061	0.691	1.023	0.380	1.191	0.402	0.673	1.023
2	1.222	0.149	0.337	0.769	0.178	0.792	0.187	0.826	0.865
3	0.874	0.632	0.176	0.983	1.032	0.926	1.027	1.026	0.983
4	0.927	0.736	1.028	0.873	0.922	1.028	0.826	0.983	0.763

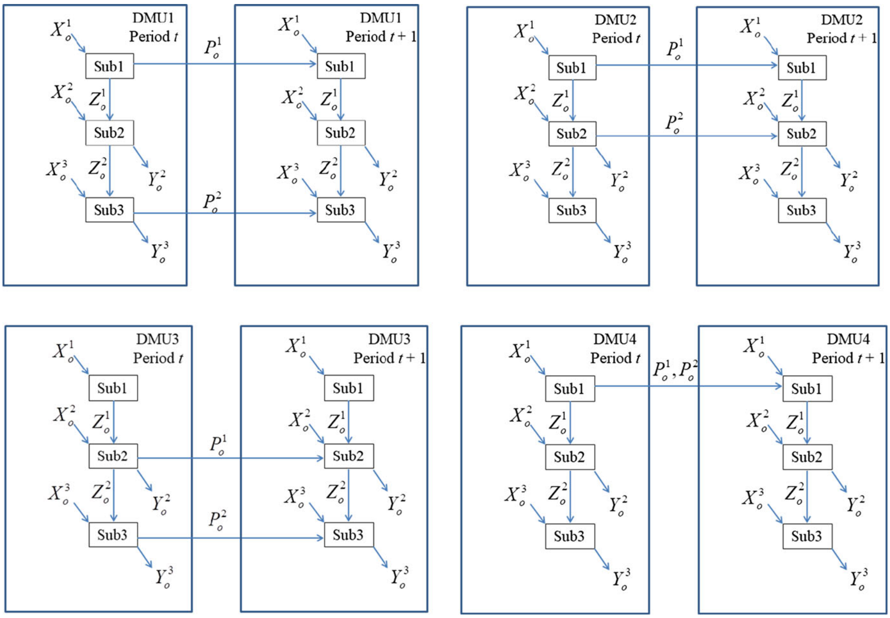

Fig. 8 The structure of DMUs in the dynamic network DEA example

Therefore, we can use the dynamic DEA model based on the sun network structure to evaluate the performance of the four DMUs. Here the weights for the three periods are set to 0.3, 0.3 and 0.4. This implies we consider the most recent period as the most important one. The resulting efficiency scores for each DMU are given in Table 7.

It is clear from Table 7 that DMUs 2, 3 and 4 are efficient. For them, their true and pseudo efficiencies are the same. DMU 1 is inefficient and its true efficiency score is

Table 7 Efficiency scores based on the dynamic DEA with the sun network structure

DMU	Pseudo efficiency score	True efficiency score
1	0.740	0.710
2	1.000	1.000
3	1.000	1.000
4	1.000	1.000

Table 8 Efficiency scores for sub-units

DMU	Sub-unit 1	Sub-unit 2	Sub-unit 3
1	0.630	0.868	0.872
2	1.000	1.000	1.000
3	1.000	1.000	1.000
4	1.000	1.000	1.000

Table 9 Efficiencies for each time period

Period	Efficiency
1	0.840
2	1.000
3	1.000

lower than pseudo efficiency score. We can calculate the efficiency scores for each sub-unit within each DMU according to (25). The results are presented in Table 8.

DMUs 2, 3 and 4 are efficient (Table 7), thus their sub-units are all efficient. DMU 1 is inefficient, and its overall efficiency score (Table 7) lies in between the highest and lowest efficiency scores of its sub-units. We can also derive the efficiency of each time period based on (18) and (19). We set that the four DMUs as equally important, which means the weights for each DMU are equal: $\varpi_1 = \varpi_2 = \varpi_3 = \varpi_4 = 0.25$. The time-specific efficiencies are given in Table 9.

Data in Table 9 suggest the second and third periods are efficient ones, whereas the first period is inefficient. The efficiency of each sub-unit in each period can be calculated based on (26) and (27). The corresponding results are presented in Table 10.

Only the sub-units of DMU1 in period 1 are inefficient while all the others are efficient. The second and third periods are efficient; therefore, all the sub-units in these two periods are efficient. The first period is inefficient. Accordingly, there must exist some sub-units in period 1 which are inefficient. And the sub-units of DMUs 2 and 3 in period 1 are all efficient. Table 10 suggests the inefficiency of period 1 stems from DMU1 and Sub-unit 1 in particular (however, the other two sub-units ae also inefficient). DMUs 2, 3 and 4 are all efficient and, as a result, all the sub-units in the three DMUs are efficient in any period.

Table 10 Efficiency of each sub-unit in each period

	Period 1	Period 2	Period 3
DMU1			
Sub-1	0.296	1	1
Sub-2	0.676	1	1
Sub-3	0.681	1	1
DMU2			
Sub-1	1	1	1
Sub-2	1	1	1
Sub-3	1	1	1
DMU3			
Sub-1	1	1	1
Sub-2	1	1	1
Sub-3	1	1	1
DMU4			
Sub-1	1	1	1
Sub-2	1	1	1
Sub-3	1	1	1

4 Conclusions

DEA is widely employed in efficiency analysis. The traditional DEA models ignore the inner structures of the DMU. The network DEA models allow modeling the inner structure, yet homogeneity in this structure is assumed. In the paper, we introduced the dummy connector to allow for heterogeneity in the underlying network structures. The dummy connector is associated with the sun network structure. As a result, different network structures can be transformed into the sun network structure relying on the dummy connector. The proposed setting requires DMUs to be homogenous in terms of the numbers of inputs, outputs and intermediate inputs.

In the static situation, the intermediate inputs work either the inputs or the outputs from the system viewpoint. This was accounted for by introducing the refined efficiency measures. In the dynamic situation, the dummy connector is introduced in every period. The inputs, outputs and intermediate inputs are then distributed thorough the connector within each period. The carry-over products are distributed between the connectors of the two adjacent time periods. There is the two-way relationship between the sub-units and the connector while there is the one-way relationship between connectors in the adjacent time periods.

We also defined the measures of efficiency for sub-units, time periods and sub-units in different time periods. However, the latter two types of efficiency scores are not uniquely determined. Future work can focus on this topic. The static and dynamic DEA models based on the sun network structure might render high numbers of efficient DMUs. Ranking of the efficient DMUs is yet another direction for further research.

Acknowledgements This research is funded by the European Social Fund according to the activity 'Improvement of researchers' qualification by implementing world-class R&D projects' of Measure No. 09.3.3-LMT-K-712. This research was supported by the 111 Project (No. B18021).

References

Barat M, Tohidi G, Sanei M (2018) DEA for nonhomogeneous mixed networks. Asia Pac Manag Rev. https://doi.org/10.1016/j.apmrv.2018.02.003

Charnes A, Cooper WW, Rhodes E (1978) Measuring the efficiency of decision making units. Eur J Oper Res 2(6):429–444

Charnes A, Cooper WW, Golany B, Halek R, Klopp G, Schmitz E, Thomas D (1986) Two-phase data envelopment analysis approaches to policy evaluation and management of army recruiting activities: tradeoffs between joint services and army advertising. Technical Report, Research Report CCS, 532

Chen Y, Zhu J (2004) Measuring information technology's indirect impact on firm performance. Inf Technol Manag 5(1–2):9–22

Chen K, Zhu J (2018) Scale efficiency in two-stage network DEA. J Oper Res Soc. https://doi.org/10.1080/01605682.2017.1421850

Chen Y, Cook WD, Li N, Zhu J (2009) Additive efficiency decomposition in two-DEA. Eur J Oper Res 196(3):1170–1176

Chen Y, Cook WD, Zhu J (2010) Deriving the DEA frontier for the two-stage process. Eur J Oper Res 202(1):138–142

Chen CL, Zhu J, Yu JY, Noori H (2012) A new methodology for evaluating sustainable product design performance with two-stage network data envelopment analysis. Eur J Oper Res 221(2):348–359

Chen Y, Cook WD, Kao C, Zhu J (2013) Network DEA pitfall: divisional efficiency and frontier projection under general network structures. Eur J Oper Res 226(3):507–515

Cook WD, Seiford LM (2011) Data envelopment analysis (DEA)—thirty years on. Eur J Oper Res 212(2):411–416

Cook WD, Liang L, Zhu J (2010) Measuring performance of two-stage network structures by DEA: a review and future perspective. Omega 38(6):423–430

Du J, Liang L, Chen Y, Cook WD, Zhu J (2011) A bargaining game model for measuring performance of two-stage network structures. Eur J Oper Res 210(2):390–397

Emrouznejad A, Parker BR, Tavares G (2008) Evaluation of research in efficiency and productivity: a survey and analysis of the first 30 years of scholar literature in DEA. Socio Econ Plan Sci 42(3):151–157

Fang ST, Ji X, Ji XH, Wu J (2018) Sustainable urbanization performance evaluation and benchmarking: an efficiency perspective. Manag Env Qual 29(2):240–254. https://doi.org/10.1108/MEQ-07-2017-0063

Färe R, Grosskopf S, Brannlund R (1996) Intertemporal production frontiers: with dynamic DEA. Kluwer Academic, Boston

Färe R, Grabowski R, Grosskopf S, Kraft S (1997) Efficiency of a fixed but allocatable input: a non-parametric approach. Econ Lett 56(2):187–193

Guo C, Shureshjani RA, Foroughi AA, Zhu J (2017) Decomposition weights and overall efficiency in two-stage additive network DEA. Eur J Oper Res 257(3):896–906

Jablonsky J (2016) Efficiency analysis in multi-period systems: an application to performance evaluation in Czech higher education. Cent Eur J Oper Res 24:283–296. https://doi.org/10.1007/s10100-015-0401-z

Jablonsky J (2018) Ranking of countries in sporting events using two-stage data envelopment analysis models: a case of Summer Olympic Games 2016. Cent Eur J Oper Res. https://doi.org/10.1007/s10100-018-0537-8

Kao C (2014) Network data envelopment analysis: a review. Eur J Oper Res 239:1–16

Kapelko M (2018) Measuring inefficiency for specific inputs using data envelopment analysis: evidence from construction industry in Spain and Portugal. Cent Eur J Oper Res 26:43–66. https://doi.org/10.1007/s10100-017-0473-z

Keshavarz Ghorabaee M, Amiri M, Olfat L, Khatami Firouzabadi SMA (2017) Designing a multi-product multi-period supply chain network with reverse logistics and multiple objectives under uncertainty. Technol Econ Dev Eco 23(3):520–548

Li H, Chen C, Cook WD, Zhang J, Zhu J (2018) Two-stage network DEA: who is the leader? Omega 74:15–19

Lim SM, Zhu J (2013) Integrated data envelopment analysis: global vs. local optimum. Eur J Oper Res 229(1):276–278

Lv KJ, Wang D, Cheng Y (2017) Measuring the dynamic performances of innovation production process from the carry-over perspective: an empirical study of China's high-tech industry. Transform Bus Econ 16(3C):345–361

Mahdiloo M, Toloo M, Duong TT, Farzipoor Saen R, Tatham P (2018) Integrated data envelopment analysis: linear vs. nonlinear model. Eur J Oper Res 268:255–267

Park S, Park KT (2009) Measurement of multiperiod aggregative efficiency. Eur J Oper Res 193(2):567–580

Seiford LM, Zhu J (1999a) Profitability and marketability of the top 55 US commercial banks. Manage Sci 45(9):1270–1288

Seiford LM, Zhu J (1999b) An investigation of returns to scale in data envelopment analysis. Omega 27:1–11

Song ML, Peng L, Wang JL, Zhao JJ (2018) Environmental efficiency and economic growth of China: a Ray slack-based model analysis. Eur J Oper Res 269(1):51–63

Toloo M, Emrouznejad A, Moreno P (2015) A linear relational DEA model to evaluate two-stage processes with shared inputs. Comput Appl Math 36(1):45–61

Toloo M, Nalchigar S, Sohrabi B (2018) Selecting most efficient information system projects in presence of user subjective opinions: a DEA approach. Cent Eur J Oper Res. https://doi.org/10.1007/s10100-018-0549-4

Tone K (2001) A slacks-based measure of efficiency in data envelopment analysis. Eur J Oper Res 130:498–509

Tone K, Tsutsui M (2009) Network DEA: a slacks-based measure approach. Eur J Oper Res 197(1):243–252

Tran CDT, Villano RA (2018) Measuring efficiency of Vietnamese public colleges: an application of the DEA-based dynamic network approach. Int Trans Oper Res 25(2):683–703

Villa G, Lozano S (2018) Dynamic network DEA approach to basketball games efficiency. J Oper Res Soc. https://doi.org/10.1080/01605682.2017.1409158

Wei QL, Chang TS (2011) Optimal system design series-network DEA models. J Oper Res Soc 62(9):1109–1119

Yin CB (2017) Environmental efficiency and its determinants in the development of China's western regions in 2000–2014. Chin J Popul Resour Environ 15(2):157–166. https://doi.org/10.1080/10042857.2017.1327687

Zhu J (2000) Multi-factor performance measure model with an application to fortune 500 companies. Eur J Oper Res 123(1):105–124